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Uniform Asymptotic Expansion for the Incomplete Beta Function

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Abstract. In [Temme N.M., Special functions. An introduction to the classical functions of mathematical physics, *A Wiley-Interscience Publication*, John Wiley & Sons, Inc., New York, 1996, Section 11.3.3.1] a uniform asymptotic expansion for the incomplete beta function was derived. It was not obvious from those results that the expansion is actually an asymptotic expansion. We derive a remainder estimate that clearly shows that the result indeed has an asymptotic property, and we also give a recurrence relation for the coefficients.

Key words: incomplete beta function; uniform asymptotic expansion

2010 Mathematics Subject Classification: 41A60; 33B20

1 Introduction

For positive real numbers a , b and $x \in [0, 1]$, the (normalised) incomplete beta function $I_x(a, b)$ is defined by

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1-t)^{b-1} dt,$$

where $B(a, b)$ denotes the ordinary beta function:

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

(see, e.g., [2, Section 8.17(i)]). In this paper, we will use the notation of [2, Section 8.18(ii)].

The incomplete beta function plays an important role in statistics in connection with the beta distribution (see, for instance, [1, pp. 210–275]). Large parameter asymptotic approximations are useful in these applications. For fixed x and b , one could use the asymptotic expansion

$$I_x(a, b) = \frac{x^a(1-x)^{b-1}}{aB(a, b)} {}_2F_1\left(\begin{matrix} 1, 1-b \\ a+1 \end{matrix}; \frac{x}{x-1}\right) \sim \frac{x^a(1-x)^{b-1}}{aB(a, b)} \sum_{n=0}^{\infty} \frac{(1-b)_n}{(a+1)_n} \left(\frac{x}{x-1}\right)^n, \quad (1)$$

as $a \rightarrow +\infty$. The right-hand side of (1) converges only for $x \in [0, \frac{1}{2})$, but for any fixed $x \in [0, 1)$ it is still useful when used as an asymptotic expansion as $a \rightarrow +\infty$. For more details, see [3, Section 11.3.3]. However, it is readily seen that (1) breaks down as $x \rightarrow 1$. Since this limit has significant importance in applications, Temme derived in [3, Section 11.3.3.1] an asymptotic expansion as $a \rightarrow +\infty$ that holds uniformly for $x \in (0, 1]$. His result can be stated as follows.

Theorem 1. *Let $\xi = -\ln x$. Then for any fixed positive integer N and fixed positive real b ,*

$$I_x(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)} \left(\sum_{n=0}^{N-1} d_n F_n + \mathcal{O}(a^{-N}) F_0 \right), \quad (2)$$

as $a \rightarrow +\infty$, uniformly for $x \in (0, 1]$. The functions $F_n = F_n(\xi, a, b)$ are defined by the recurrence relation

$$aF_{n+1} = (n+b-a\xi)F_n + n\xi F_{n-1}, \quad (3)$$

with

$$F_0 = a^{-b} Q(b, a\xi), \quad F_1 = \frac{b-a\xi}{a} F_0 + \frac{\xi^b e^{-a\xi}}{a\Gamma(b)},$$

and $Q(a, z) = \Gamma(a, z)/\Gamma(a)$ is the normalised incomplete gamma function (see [2, Section 8.2(i)]). The coefficients $d_n = d_n(\xi, b)$ are defined by the generating function

$$\left(\frac{1-e^{-t}}{t} \right)^{b-1} = \sum_{n=0}^{\infty} d_n (t-\xi)^n. \quad (4)$$

In particular,

$$d_0 = \left(\frac{1-x}{\xi} \right)^{b-1}, \quad d_1 = \frac{x\xi + x - 1}{(1-x)\xi} (b-1) d_0.$$

They satisfy the recurrence relation

$$\begin{aligned} \xi(n+1)(n+2)d_0 d_{n+2} &= \xi \sum_{m=0}^n (m+1) \left(n-2m+1 + \frac{m-n-1}{b-1} \right) d_{m+1} d_{n-m+1} \\ &\quad + \sum_{m=0}^n (m+1) \left(n-2m-2-\xi + \frac{m-n}{b-1} \right) d_{m+1} d_{n-m} \\ &\quad + \sum_{m=0}^n (1-m-b) d_m d_{n-m}. \end{aligned} \quad (5)$$

In the case that $b = 1$, we have $d_0 = 1$ and $d_n = 0$ for $n \geq 1$.

Our contribution is the remainder estimate in (2) and the recurrence relation (5). In fact, it is not at all obvious from (3) that the sequence $\{F_n\}_{n=0}^{\infty}$ has an asymptotic property as $a \rightarrow +\infty$. We will show that for any non-negative integer n ,

$$0 < F_{n+1} \leq \frac{n+\beta}{a} F_n, \quad (6)$$

where $\beta = \max(1, b)$.

In [4, Section 38.2.8] the function F_n is identified as a Kummer U -function:

$$F_n = \frac{\xi^{n+b} e^{-a\xi} n!}{\Gamma(b)} U(n+1, n+b+1, a\xi).$$

2 Proof of the main results

We proceed similarly as in [3, Section 11.3.3.1] and start with the integral representation

$$I_x(a, b) = \frac{1}{B(a, b)} \int_{\xi}^{+\infty} t^{b-1} e^{-at} \left(\frac{1 - e^{-t}}{t} \right)^{b-1} dt. \quad (7)$$

We substitute the truncated Taylor series expansion

$$\left(\frac{1 - e^{-t}}{t} \right)^{b-1} = \sum_{n=0}^{N-1} d_n (t - \xi)^n + r_N(t)$$

into (7) and obtain

$$I_x(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)} \left(\sum_{n=0}^{N-1} d_n F_n + R_N(a, b, x) \right),$$

where F_n is given by the integral representation

$$F_n = \frac{1}{\Gamma(b)} \int_{\xi}^{+\infty} t^{b-1} e^{-at} (t - \xi)^n dt = \frac{e^{-a\xi}}{\Gamma(b)} \int_0^{+\infty} (\tau + \xi)^{b-1} \tau^n e^{-a\tau} d\tau, \quad (8)$$

and the remainder term $R_N(a, b, x)$ is defined by

$$R_N(a, b, x) = \frac{1}{\Gamma(b)} \int_{\xi}^{+\infty} t^{b-1} e^{-at} r_N(t) dt. \quad (9)$$

The recurrence relation (3) can be obtained from (8) via a simple integration by parts.

Let, for a moment,

$$c_n(a, b) = \int_0^{+\infty} (\tau + \xi)^{b-1} \tau^n e^{-a\tau} d\tau.$$

Then via integration by parts we find

$$ac_{n+1}(a, b) = (n+b)c_n(a, b) + \xi(1-b)c_n(a, b-1). \quad (10)$$

We make the observation that

$$0 \leq \xi c_n(a, b-1) = \xi \int_0^{+\infty} (\tau + \xi)^{b-2} \tau^n e^{-a\tau} d\tau \leq c_n(a, b). \quad (11)$$

It follows from (10) and (11) that

$$ac_{n+1}(a, b) \leq \begin{cases} (n+1)c_n(a, b) & \text{if } 0 < b \leq 1, \\ (n+b)c_n(a, b) & \text{if } b \geq 1. \end{cases}$$

Since $F_n = e^{-a\xi} c_n(a, b) / \Gamma(b)$, this inequality implies (6).

To obtain the remainder estimate in (2), we use the Cauchy integral representation

$$r_N(t) = \frac{(t - \xi)^N}{2\pi i} \oint_{\{\xi, t\}} \frac{\left(\frac{1 - e^{-\tau}}{\tau} \right)^{b-1}}{(\tau - t)(\tau - \xi)^N} d\tau, \quad (12)$$

where the contour encircles the points ξ and t once in the positive sense. From the integral representation (9), we have that $0 \leq \xi \leq t$. Thus, in the case that $N \geq 1$, we can deform the contour in (12) to the path

$$[1 + \infty i, 1 + \pi i] \cup [1 + \pi i, -1 + \pi i] \cup [-1 + \pi i, -1 - \pi i] \\ \cup [-1 - \pi i, 1 - \pi i] \cup [1 - \pi i, 1 - \infty i].$$

For the integrals along the final three portions of the path, we have the estimates

$$\left| \frac{1}{2\pi i} \int_{-1-\pi i}^{-1-\pi i} \frac{\left(\frac{1-e^{-\tau}}{\tau}\right)^{b-1}}{(\tau-t)(\tau-\xi)^N} d\tau \right| \leq \frac{\max\left((e-1)^{b-1}, \left(\frac{e+1}{\sqrt{\pi^2+1}}\right)^{b-1}\right)}{(1+\xi)^{N+1}},$$

$$\left| \frac{1}{2\pi i} \int_{-1-\pi i}^{1-\pi i} \frac{\left(\frac{1-e^{-\tau}}{\tau}\right)^{b-1}}{(\tau-t)(\tau-\xi)^N} d\tau \right| \leq \frac{\max\left(\left(\frac{e^{\pm 1}+1}{\sqrt{\pi^2+1}}\right)^{b-1}\right)}{\pi^{N+2}}, \quad (13)$$

and

$$\left| \frac{1}{2\pi i} \int_{1-\pi i}^{1-\infty i} \frac{\left(\frac{1-e^{-\tau}}{\tau}\right)^{b-1}}{(\tau-t)(\tau-\xi)^N} d\tau \right| \leq \frac{1}{2\pi} \int_{\pi}^{+\infty} \frac{\max\left((1 \pm e^{-1})^{b-1}\right)(s^2+1)^{(1-b)/2}}{\sqrt{s^2+(1-t)^2}(s^2+(1-\xi)^2)^{N/2}} ds$$

$$\leq \frac{\max\left((1 \pm e^{-1})^{b-1}\right)}{2\pi} \int_{\pi}^{+\infty} \frac{(s^2+1)^{(1-b)/2}}{s^{N+1}} ds, \quad (14)$$

respectively. The integrals along the first two portions can be estimated similarly to (13) and (14). Hence, for $0 \leq \xi \leq t$ and $N \geq 1$, we have

$$|r_N(t)| \leq C_N(b)(t-\xi)^N,$$

where the constant $C_N(b)$ does not depend on ξ . Using this result in the integral representation (9), we can infer that

$$|R_N(a, b, x)| \leq C_N(b)F_N.$$

Finally, combining this result with the inequalities (6), we obtain the required remainder estimate in (2).

The reader can check that the function $f(t) = \left(\frac{1-e^{-t}}{t}\right)^{b-1}$ is a solution of the nonlinear differential equation

$$tf(t)f''(t) - \frac{b-2}{b-1}tf'^2(t) + (t+2)f(t)f'(t) + (b-1)f^2(t) = 0.$$

If we substitute the Taylor series (4) into this differential equation and rearrange the result, we obtain the recurrence relation (5).

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